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An Identity for Spline Functions with Applications to Variation-Diminishing Spline Approximation[1]

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INTRODUCTION

I. J. Schoenberg [22] has recently constructed a generalization of Bernstein polynomial approximation, which associates to functions $f(x)$, defined on [0, 1], the approximation

$$S_\Delta^k f(x) = \sum_{j=-k}^{n-1} f(\xi_j)\, N_j(x) \qquad (n > 0, k > 0). \tag{1}$$

$S_\Delta^k f(x)$ is a spline function of degree k, having knots from Δ, where

$$\Delta = \{x_i\}_0^n \qquad (0 = x_0 < x_n < \cdots < x_n = 1).$$

The "nodes" ξ_j and the "fundamental functions" $N_j(x)$ depend on k and Δ.

The approximations (1) *reproduce linear functions* and *are variation-diminishing* (see [22]). We shall refer to them as "variation-diminishing spline approximations." They have the shape-preserving properties of Bernstein polynomials and, when appropriately selected, converge much faster than Bernstein polynomials (See [22], and Marsden and Schoenberg [16]).

In part I, we shall prove a generalization of the Weierstrass approximation theorem, by showing that (1) converges uniformly to $f(x)$ for continuous functions, if and only if

$$\frac{\|\Delta\|}{k} \to 0,$$

where, as usual, $\|\Delta\| = \max_j(x_{j+1} - x_j)$.

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The primary tool will be an identity relating powers and B-splines. My original proof of this identity being quite cumbersome, I give, instead, a proof supplied by T. N. E. Greville (private communication).

In Part II we make the necessary, but slight, alterations in Greville's proof, which yield the corresponding identity for Tchebycheff spline functions. From this identity, "Tchebycheffian spline approximation," extending Schoenberg's spline approximation, is defined. A convergence theorem for Tchebycheffian spline approximation concludes Part II.

In Part III, Schoenberg's variation-diminishing spline approximations are discussed in more detail, with emphasis on their similarities to (and differences from) Bernstein polynomials. Their potential for application is also discussed. Topics covered include convergence of derivatives, approximation to convex functions, and constraints on the nodes.

In Part IV, we discuss the possibility of extending Voronovskaya's theorem and related asymptotic formulae.

Part I: The Weierstrass Approximation Theorem for Spline Functions

1. *Divided Differences and B-Splines*

The purpose of this section is twofold: (1) to recall the facts needed in the proof of Theorem 1 below, and (2) to make the discussion of Tchebycheff splines in Part II more lucid. For this latter reason, more details than necessary are included. There is no new material.

For a given function $f(t)$ and points $t_1, t_2, \ldots, t_{k+1}$ $(k \geqslant 0)$, the *divided difference* $f(t_1, t_2, \ldots, t_{k+1})$ is defined by

$f(t_1, t_2, \ldots, t_{k+1}) =$ the coefficient of t^k in the unique polynomial of degree k or less which interpolates $f(t)$ at $t_1, t_2, \ldots, t_{k+1}$.

Multiple values are permitted among the points t_i. For such multiple values we interpolate one or more derivatives of $f(t)$ as well as $f(t)$ itself.

In terms of divided differences and the *Newton polynomials*

$$(t - t_1)(t - t_2) \cdots (t - t_k) \qquad (k > 0),$$

Hermite interpolation is specified by

$$f(t) = f(t_1) + \sum_{k=1}^{k=m} (t - t_1) \cdots (t - t_k) f(t_1, \ldots, t_{k+1}) + R(t), \tag{1.1}$$

where $m \geqslant 0$, and $R(t)$ is the remainder, given by

$$R(t) = (t - t_1) \cdots (t - t_{m+1}) f(t_1, \ldots, t_{m+1}, t). \tag{1.2}$$

Writing (1.1) with t_{m+1} replaced by t, then with t_{m+1} replaced by t', and subtracting yields the *recurrence relation*

$$(t - t_1) \cdots (t - t_m)[f(t_1, ..., t_m, t) - f(t_1, ..., t_m, t')]$$
$$= (t - t_1) \cdots (t - t_m)(t - t') f(t_1, ..., t_m, t, t') \qquad (1.3)$$

which can, of course, be simplified.

Spline functions will now be defined. Let m be a positive integer and $\{x_j\}$ a doubly infinite sequence of real numbers satisfying

$$x_{j-1} \leqslant x_j < x_{j+m} \qquad (-\infty < j < +\infty). \qquad (1.4)$$

We denote $\lim\inf x_j$ by α and $\lim\sup x_j$ by β, usually supposing that both are infinite.

By a *spline function* of order m, or degree $m - 1$, having the knots x_j, we shall mean a real function $s(x)$ satisfying:

(a) $s(x)$ is a polynomial of degree $m - 1$ or less on $(x_j, x_{j+1}]$ whenever $x_j < x_{j+1}$; and

(b) $s(x)$ has a continuous $(m - i)$th derivative on (x_j, x_{j+i}) whenever $x_j < x_{j+i}$. (1.5)

On (α, β), a basis for the set of spline functions is provided by the B-splines $M_j(x)$, which may be defined by

$$M_j(x) = M(x; x_j, x_{j+1}, ..., x_{j+m}) \qquad (-\infty < j < +\infty), \qquad (1.6)$$

where

$$M(x; t) = m(t - x)_+^{m-1} = m(t - x)^{m-1} \quad \text{if } t \geqslant x,$$
$$= 0 \quad \text{if } t < x; \qquad (1.7)$$

i.e., for a given x, $M_j(x)$ is the divided difference of $M(x; t)$ over the knots $t = x_j, ..., x_{j+m}$.

The term $M_j(x)$ is a nonnegative spline function which vanishes outside $(x_j, x_{j+m}]$.

The B-splines are discussed in detail in Curry and Schoenberg [5]. (See also [22].)

2. *The B-Spline Representation of Polynomials*

It has long been known (see [5]) that every polynomial of degree at most $m - 1$ has a unique representation on (α, β) as a linear combination of the B-splines $M_j(x)$.

The following theorem gives the information necessary to specify these representations explicitly.

THEOREM 1. *Let p, q be integers such that*

$$x_{p+m-1} < x_{q+1} .$$

Then the relation

$$m(t-x)^{m-1} = \sum_{j=p}^{j=q} (x_{j+m} - x_j)(t - x_{j+1}) \cdots (t - x_{j+m-1}) M_j(x), \tag{2.1}$$

is valid for all complex t and for $x_{p+m-1} < x \leqslant x_{q+1}$.

Proof. For fixed x, let

$R_j(t) = M(x; t)$—the polynomial in t of degree m or less which interpolates $M(x; t)$ at $t = x_j ,..., x_{j+m}$.

From (1.2),

$$R_j(t) = (t - x_j) \cdots (t - x_{j+m})\, M(x; x_j ,..., x_{j+m} , t) \tag{2.2}$$

and, hence,

$$R_{j+1}(t) = (t - x_{j+1}) \cdots (t - x_{j+m+1})\, M(x; x_{j+1} ,..., x_{j+m+1} , t). \tag{2.3}$$

Using (1.3), with $t_i = x_{j+i}$ $(1 \leqslant i \leqslant m)$ and with $t' = x_j$, in (2.2), then with $t' = x_{j+m+1}$ in (2.3), and subtracting yields

$$R_j(t) - R_{j+1}(t) = (t - x_{j+1}) \cdots (t - x_{j+m})[M_{j+1}(x) - M_j(x)].$$

Hence, for $p \leqslant q$,

$$R_{p-1}(t) - R_{q+1}(t) = \sum_{j=p-1}^{j=q} (t - x_{j+1}) \cdots (t - x_{j+m})[M_{j+1}(x) - M_j(x)]. \tag{2.4}$$

Now, for $x \leqslant x_{q+1}$,

$$M_{q+1}(x) = 0 \qquad \text{and} \qquad R_{q+1}(t) = M(x; t) - m(t-x)^{m-1}. \tag{2.5}$$

Similarly, for $x_{p+m-1} < x$,

$$M_{p-1}(x) = 0 \qquad \text{and} \qquad R_{p-1}(t) = M(x; t). \tag{2.6}$$

Substitution of (2.5) and (2.6) into (2.4) yields the required result (2.1), after some reindexing and cancellation.

As mentioned previously, the above proof is due to T.N.E. Greville.

By letting $p + m - 1 = 0$ and $q + 1 = n$, we have the following:

COROLLARY. *Let* $n > 0$, $x_0 = a$, $x_n = b$. *Let* $m \geqslant 3$,

$$N_j(x) = \frac{x_{j+m} - x_j}{m} M_j(x) \qquad (-m < j < n), \tag{2.7}$$

$$\xi_j = \frac{x_{j+1} + \cdots + x_{j+m-1}}{m-1} \qquad (-m < j < n), \tag{2.8}$$

and

$$\xi_j^{(2)} = \frac{x_{j+1}x_{j+2} + \cdots + x_{j+m-2}x_{j+m-1}}{\binom{m-1}{2}} \qquad (-m < j < n). \tag{2.9}$$

Then, for $a \leqslant x \leqslant b$,

$$1 = \sum_{1-m}^{n-1} N_j(x), \tag{2.10}$$

$$x = \sum_{1-m}^{n-1} \xi_j N_j(x), \tag{2.11}$$

and

$$x^2 = \sum_{1-m}^{n-1} \xi_j^{(2)} N_j(x), \tag{2.12}$$

provided that $N_{1-m}(a)$ *be replaced by* $N_{1-m}(a+)$ *whenever* $x_{1-m} = x_0 = a$.

T. N. E. Greville first described (2.7), (2.8), (2.10), and (2.11) in his supplement to Schoenberg's paper [22]. Here the relations (2.10–12) result from equating the coefficients of $t^{m-1}, t^{m-2}, t^{m-3}$ in (2.1). The new result here, (2.12), will be useful in establishing convergence theorems about the approximation method developed by Schoenberg in [22] (and redeveloped in Section 3 below).

The above-mentioned modification in the definition of $N_{1-m}(a)$ is sufficient (and necessary) to make (2.1) valid at the left endpoint of $[a, b]$ (see [5], page 79). Observe that (2.10) and (2.11) are valid for $m = 2$, also.

3. *Spline Approximation on* $[a, b]$

Let $n > 0$, $k > 0$ be integers and let

$$\Delta = \{x_i\}_0^n$$

be a sequence of real numbers satisfying

$$a = x_0 < x_1 \leqslant x_2 \leqslant \cdots < x_n = b$$

and (3.1)

$$x_{i-k} < x_i \qquad (k < i < n).$$

We extend the sequence by setting

$$x_{-k} = x_{-k+1} = \cdots = x_{-1} = a \qquad \text{and} \qquad x_{n+1} = \cdots = x_{n+k} = b$$

and let $N_j(x)$ and ξ_j $(-k \leqslant j < n)$ be defined by (2.7) and (2.8), with $k = m - 1$. Then,

$$a = \xi_{-k} < \xi_{-k+1} < \cdots < \xi_{n-1} = b. \tag{3.2}$$

The relations (2.10), (2.11), and (3.2) suggest the following approximation method.

To each $f(x)$ defined on $[a, b]$, associate the spline function

$$Sf(x) = S_\Delta{}^k f(x) = \sum_{j=-k}^{n-1} f(\xi_j) N_j(x) \qquad (a \leqslant x \leqslant b). \tag{3.3}$$

$Sf(x)$ is to be regarded as an approximation to $f(x)$ on $[a, b]$.

This approximation method was introduced by I. J. Schoenberg in [22]. The "nodes" ξ_j and the "fundamental functions" $N_j(x)$ are determined by the degree k and the knots Δ. It is easily seen that

$$Sf(x) \in C[a, b],$$

$$Sf(a) = f(a), \qquad Sf(b) = f(b),$$

and that, when the inequalities of (3.1) are all strict,

$$Sf(x) \in C^{k-1}[a, b].$$

I. J. Schoenberg has proved the following important theorem (see [22]).

Schoenberg's Theorem. The approximation $Sf(x)$ to $f(x)$ is exact for linear functions and is variation-diminishing. This means that for every linear function

$$l(x) \equiv cx + d,$$

the difference $Sf(x) - l(x)$ has no more variations in sign on the interval $[a, b]$ than the difference $f(x) - l(x)$ has.

Because of this theorem, we shall refer to (3.3) as "variation-diminishing spline approximation."

A consequence of Schoenberg's theorem is that

$$\textit{If } f(x) \geqslant 0 \textit{ on } [a, b], \textit{ then } Sf(x) \geqslant 0 \textit{ on } [a, b]. \tag{3.4}$$

Combining this result with the well-known Bohman–Korovkin theorem (see [4], [14]) yields:

THEOREM 2. *A necessary and sufficient condition that*

$$\lim Sf(x) = f(x) \qquad \textit{uniformly in } [a, b], \tag{3.5}$$

for every function $f(x) \in C[a, b]$, is that (3.5) hold for the particular function $f(x) \equiv x^2$.

By "$\lim Sf(x)$" is meant the limit of the sequence of approximations corresponding to a sequence of values of Δ and k.

Proof. Because of (3.4), the Bohman–Korovkin theorem applies, namely, a necessary and sufficient condition that (3.5) hold for every continuous function is that it hold for the particular functions $f(x) \equiv 1$, $f(x) \equiv x$, $f(x) \equiv x^2$. From Schoenberg's theorem (or (2.10), (2.11)), Eq. (3.5) is trivially satisfied for the first two of these functions.

4. *The Weierstrass Approximation Theorem for Spline Functions*

In this section, Theorem 2 and the Corollary to Theorem 1 are used to develop necessary and sufficient conditions on Δ and k that variation-diminishing spline approximations to a continuous function converge uniformly. We then use this result to state an extension of the Weierstrass approximation theorem.

LEMMA. *Let $E(x)$ be the error in approximating the function $g(x) = x^2$ by $Sg(x)$ of* (3.3); i.e.,

$$E(x) = Sg(x) - g(x).$$

Then, for $a \leqslant x \leqslant b$,

$$0 \leqslant E(x) \leqslant \frac{1}{2k} \max_{0 \leqslant j \leqslant n} (x_{j+k} - x_j)^2 \leqslant \min \left\{ \frac{(b-a)^2}{2k}, \frac{k \| \Delta \|^2}{2} \right\}; \tag{4.1}$$

if $k = 1$,

$$\max E(x) = \tfrac{1}{4} \| \Delta \|^2; \tag{4.2}$$

if $n = 1$,

$$E(x) = \frac{1}{k} \left(\frac{x-a}{b-a} \right) \left(\frac{b-x}{b-a} \right); \tag{4.3}$$

and if $k \geqslant 2$,

$$E(x) = \sum_{-k}^{n-1} \lambda_j N_j(x), \tag{4.4}$$

where

$$\lambda_j = \frac{1}{k^2(k-1)} \sum_{j+1 \leqslant r \leqslant s \leqslant j+k} (x_s - x_r)^2. \tag{4.5}$$

Proof. For $k = 1$, the approximation method (3.3) is equivalent to linear interpolation. For $n = 1$, (3.3) gives Bernstein polynomial approximation on $[a, b]$, (see [22], [16] for details). These facts imply (4.2) and (4.3), respectively.

To prove (4.4), we note that for $g(x) \equiv x^2$,

$$Sg(x) = \sum_{-k}^{n-1} (\xi_j)^2 N_j(x),$$

and

$$g(x) = \sum_{-k}^{n-1} \xi_j^{(2)} N_j(x)$$

where ξ_j and $\xi_j^{(2)}$ are given by (2.8), and (2.9). Thus,

$$E(x) = \sum_{-k}^{n-1} [(\xi_j)^2 - \xi_j^{(2)}] N_j(x),$$

which becomes (4.4) if we let

$$\lambda_j = (\xi_j)^2 - \xi_j^{(2)}. \tag{4.6}$$

Expansion of (4.5) and (4.6) gives identical expressions, completing the verification of (4.4).

That $E(x) \geqslant 0$ is implied by Schoenberg's theorem. The remainder of (4.1) will follow from (4.2) and (4.4). If $k = 1$, (4.1) is a restatement of (4.2) with $\frac{1}{4}$ replaced by the larger $\frac{1}{2}$. If $k \geqslant 2$, (4.4) implies

$$E(x) \leqslant \max_i \lambda_i \sum_j N_j(x) = \max_i \lambda_i .$$

Because the x_i are monotone and (4.5) involves $k(k-1)/2$ terms, for each j,

$$\lambda_j \leqslant \frac{(x_{j+k} - x_j)^2}{2k} .$$

In addition,

$$\max_j (x_{j+k} - x_j)^2 \leqslant k^2 \| \Delta \|^2,$$

and

$$\max_j (x_{j+k} - x_j)^2 \leqslant (b - a)^2,$$

so that the proof of (4.1) and, hence, of the lemma is complete.

From this lemma and Theorem 2, we get the following

THEOREM 3. *A necessary and sufficient condition that*

$$\lim Sf(x) = f(x), \qquad \textit{uniformly in } [a, b], \tag{4.7}$$

for every $f(x) \in C[a, b]$, *is that*

$$\lim \frac{\| \Delta \|}{k} = 0. \tag{4.8}$$

Proof. To prove necessity, we note that

$$\frac{\|\Delta\|}{k} = \max_j \frac{x_{j+1} - x_j}{k} \leqslant \max_j \frac{x_{j+k} - x_j}{k} = \max_j (\xi_j - \xi_{j-1}).$$

Bohman has shown in [4], pp. 43–45, that, in order for (4.7) to hold, it is necessary that the right-hand member tend to zero. Hence, (4.8) is necessary.

To prove sufficiency, we use the lemma to show that

$$\lim E(x) = 0, \qquad \text{uniformly in } [a, b]. \tag{4.9}$$

Theorem 2 will then imply that (4.7) must follow.

From the latter inequality of (4.1), (4.9) will hold if either

$$\lim k = \infty, \tag{4.10}$$

or

$$k \text{ is bounded and } \lim \|\Delta\| = 0. \tag{4.11}$$

The other possibilities encompassed by (4.8) can be handled by a simple argument. In fact, given $\epsilon > 0$, either

$$k > \frac{(b-a)^2}{2\epsilon}$$

or

$$k \leqslant \frac{(b-a)^2}{2\epsilon} \qquad \text{and} \qquad \|\Delta\| < \frac{2\epsilon}{b-a}$$

will guarantee that $\max |E(x)| < \epsilon$.

In practice, either (4.10) or (4.11) will occur. Indeed, since the "input" to the approximation (3.3) consists of $n + k$ pieces of data (the values of $f(x)$ at $x = \xi_j$, $-k \leqslant j < n$), (4.1) suggests that for a given number

$$l = n + k,$$

it may be preferable to select n large ($\|\Delta\|$ small) rather than k large. More evidence on this matter has been given in [16].

Theorem 3 is a powerful tool. For example, Theorem 2 in [16] follows from it immediately as a corollary. We shall now use Theorem 3 to prove an extension of the Weierstrass approximation theorem.

Let Δ satisfy (3.1) as extended in the sentence following (3.1) and let $\mathscr{S}$ denote the entire class of spline functions defined on $[a, b]$ of degree $k > 0$, having the knots Δ. Using the uniform norm

$$\|f\| = \max_{x\in[a,b]} |f(x)|$$

for functions $f(x)$, continuous on $[a, b]$, we define the distance of such a function from the class $\mathscr{S}$ by

$$d(f, \mathscr{S}) = \inf_{\sigma \in \mathscr{S}} \| f - \sigma \|.$$

THEOREM 4. *Let* $g(x) \in C[a, b]$. *To every* $\epsilon > 0$ *there corresponds a* $\delta = \delta_g > 0$ *such that*

$$d(g, \mathscr{S}) < \epsilon \tag{4.12}$$

whenever

$$\frac{\| \Delta \|}{k} < \delta. \tag{4.13}$$

Theorem 4 is implied by Theorem 3, since $Sg(x) \in \mathscr{S}$. If $n = 1$, $\mathscr{S}$ becomes the set of polynomials of degree at most k, while the left side of (4.13) reduces to $(b - a)/k$. Thus, Theorem 4 is an extension of Weierstrass' theorem.

We shall now leave Schoenberg's variation diminishing spline approximations and, in Part II, prove some results similar to those in Sections 2–3 above for spline functions based on Tchebycheff polynomials. We shall return to the present topic in Parts III–IV.

PART II: TCHEBYCHEFFIAN SPLINE APPROXIMATION

5. *Extended Complete Tchebycheff Systems*

Let m be a positive integer and let $w_i(x)$ $(1 \leqslant i \leqslant m)$ be real functions satisfying

$$w_i(x) \in C^{m+1}(-\infty, +\infty) \quad \text{and} \quad \operatorname*{g.l.b.}_{-\infty < x < \infty} w_i(x) > 0.$$

To this set of functions we adjoin the functions

$$w_0(x) = w_{-1}(x) = 1$$

and then define the two systems $\{u_i(x)\}_0^{m-1}$ and $\{v_i(t)\}_0^{m+1}$ by

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= \int_0^x w_1(\xi_1)\, d\xi_1, \\ u_k(x) &= \int_0^x w_1(\xi_1) \int_0^{\xi_1} w_2(\xi_2) \cdots \int_0^{\xi_{k-1}} w_k(\xi_k)\, d\xi_k \cdots d\xi_1 \qquad (1 < k < m), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
v_0(t) &= w_m(t),\\
v_1(t) &= w_m(t) \int_0^t w_{m-1}(\xi_1)\, d\xi_1 ,\\
v_k(t) &= w_m(t) \int_0^t w_{m-1}(\xi_1) \int_0^{\xi_1} w_{m-2}(\xi_2)\\
&\qquad \cdots \int_0^{\xi_{k-1}} w_{m-k}(\xi_k)\, \delta\xi_k \cdots d\xi_1 \qquad (1 < k \leqslant m+1).
\end{aligned} \tag{5.2}$$

We shall have only slight need for the functions $v_m(t)$ and $v_{m+1}(t)$. Indeed, a somewhat longer proof of Theorem 7 below avoids the need for $v_{m+1}(t)$ entirely.

The systems $\{u_i(x)\}_0^{m-1}$ and $\{v_i(t)\}_0^{m+1}$ are each *extended complete Tchebycheff* (ECT) *systems*. These have been discussed in detail in Karlin and Studden [12], especially Chapter XI. See also Karlin [10], Karlin and Schumaker [11], and Karlin and Ziegler [13].

Our notation will generally correspond to that in [12], except for $v_i(t)$.

The ECT-systems have unique interpolation properties similar to those of polynomials. The word "extended" refers to the fact that the order of interpolation at a point may be multiple, i.e., involving interpolation to several derivatives as well as to the function itself. The word "complete" refers to the fact that $\{v_i(t)\}_0^k$ is an extended Tchebycheff system for $0 \leqslant k \leqslant m+1$.

By a *v-polynomial of degree* j $(0 \leqslant j \leqslant m+1)$, we shall mean a function $v(t)$ of the form

$$v(t) = \sum_{i=0}^{i=j} a_i v_i(t) \qquad (a_i \text{ real}, a_j \neq 0).$$

We similarly define u-polynomials of degree j, $(0 \leqslant j \leqslant m-1)$.

Following [12], we define

$$V^* \begin{pmatrix} 0 & \cdots & k \\ t_0 & \cdots & t_k \end{pmatrix} = \det \| v_i(t_j) \|_{i,j=0}^k \qquad (0 \leqslant k \leqslant m+1), \tag{5.3}$$

where, for repeated t_j's, the corresponding columns of the determinant are replaced by derivative columns. For example,

$$V^* \begin{pmatrix} 0 & 1 \\ t_1 & t_1 \end{pmatrix} = \begin{vmatrix} v_0(t_1) & v_0'(t_1) \\ v_1(t_1) & v_1'(t_1) \end{vmatrix}.$$

The determinant (5.3) is never zero, and, if $t_0 \leqslant \cdots \leqslant t_k$, (5.3) is positive. See [12], p. 6.

We think of

$$v(t) = V^* \begin{pmatrix} 0 & 1 & \cdots & k \\ t & t_1 & \cdots & t_k \end{pmatrix}, \tag{5.4}$$

as being defined by (5.3) with $t = t_0 \neq t_i$ $(1 \leqslant i \leqslant k)$, then extended continuously for $t = t_i$ $(1 \leqslant i \leqslant k)$. Thus, $v(t)$ in (5.4) is a v-polynomial of degree k, with zeros at $t = t_i$ $(1 \leqslant i \leqslant k)$. If t_j occurs $r + 1$ times among the t_i's, then each derivative of $v(t)$, up through the rth, also has a zero at $t = t_j$.

Observe that, for example,

$$V^* \begin{pmatrix} 0 & 1 \\ t & t_1 \end{pmatrix} \Bigg|_{t=t_1} = 0 \neq V^* \begin{pmatrix} 0 & 1 \\ t_1 & t_1 \end{pmatrix}.$$

If (5.4) is expanded by minors about the first column, we obtain

$$v(t) = \sum_{i=0}^{j=k} (-1)^i V^* \begin{pmatrix} 0 & \cdots & \hat{\imath} & \cdots & k \\ t_1 & \cdots & t_i & \cdots & t_k \end{pmatrix} v_i(t),$$

where $\hat{\imath}$ indicates the deleted ith row.

Dividing throughout by the coefficient of $v_k(t)$ normalizes this v-polynomial to the *Newton v-polynomial* (5.5) of degree k, having the k zeros $t_1, t_2, \ldots, t_k$:

$$N^* \begin{pmatrix} 0 & 1 & \cdots & k \\ t & t_1 & \cdots & t_k \end{pmatrix} = (-1)^k \frac{V^* \begin{pmatrix} 0 & 1 & \cdots & k \\ t & t_1 & \cdots & t_k \end{pmatrix}}{V^* \begin{pmatrix} 0 & \cdots & k-1 \\ t_1 & \cdots & t_k \end{pmatrix}}. \tag{5.5}$$

In (5.5), the coefficient of $v_k(t)$ is unity, as was the case with the coefficient of t^k in the Newton polynomials of Section 1 above. However, $v_k(t)$ will correspond to $m\binom{m-1}{k} t^k$ rather than to t^k, so that Newton v-polynomials are not a direct extension of Newton polynomials. See also the comments leading up to (5.7) below.

Any v-polynomial of degree k which has k specific zeros is unique up to a multiplicative constant.

The systems $\{u_i(x)\}_0^{m-1}$ and $\{v_i(t)\}_0^{m-1}$ span the respective null spaces of the adjoint operators L and L^* given by

$$\begin{aligned} L(u) &= D_m D_{m-1} \cdots D_1(u), \\ L^*(v) &= (-1)^m D_0 D_1 \cdots D_{m-1}(v/w_m), \end{aligned} \tag{5.6}$$

where

$$D_i = \frac{1}{w_i(x)} \frac{d}{dx} \qquad (0 \leqslant i \leqslant m),$$

(See Ince [9], p. 125, and [12], p. 440).

For $w_i(x) = i$, these systems become

$$u_i(x) = x^i \qquad (0 \leqslant i < m),$$

and

$$v_i(t) = m \binom{m-1}{i} t^i \qquad (0 \leqslant i < m).$$

From the binomial theorem, we have

$$m(t-x)^{m-1} = \sum_{i=0}^{m-1} m \binom{m-1}{m-1-i} t^{m-1-i}(-x)^i,$$

or

$$N^* \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ t & x & \cdots & x \end{pmatrix} = \sum_{i=0}^{m-1} (-1)^i u_i(x)\, v_{m-1-i}(t). \tag{5.7}$$

This relation is also true in the general case, since it is easy to verify, using (5.6), that

$$N^* \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ t & x & \cdots & x \end{pmatrix}$$

$$= w_m(t) \int_x^t w_{m-1}(\xi_1) \cdots \int_x^{\xi_{m-2}} w_1(\xi_{m-1})\, d\xi_{m-1} \cdots d\xi_1 \tag{5.8}$$

and the right side of (5.8) may be transformed, by repeatedly using (5.1) and (5.2), into the right side of (5.7) (see also [12], p. 448).

We shall dignify (5.7) by calling it the *binomial theorem.*

6. *Divided v-Differences and Tchebycheffian B-Splines*

This section is an extension of Section 1 above.

For a given function $f(t)$ and points $t_1, \dots, t_{k+1}$ $(0 \leqslant k \leqslant m+1)$, the divided v-difference $f_v(t_1, \dots, t_{k+1})$ is defined by

$f_v(t_1, \dots, t_{k+1})$ = the coefficient of $v_k(t)$ in the unique v-polynomial of degree k or less which interpolates $f(t)$ at $t_1, \dots, t_{k+1}$. (6.1)

In terms of divided v-differences and the Newton v-polynomials defined by (5.5), *generalized Hermite interpolation* is specified by

$$f(t) = \sum_{k=0}^{k=m} N^* \begin{pmatrix} 0 & 1 & \cdots & k \\ t & t_1 & \cdots & t_k \end{pmatrix} f_v(t_1, \dots, t_{k+1}) + R(t), \tag{6.2}$$

where $m \geqslant 0$ and $R(t)$ is the remainder, given by

$$R(t) = N^* \begin{pmatrix} 0 & 1 & \cdots & m+1 \\ t & t_1 & \cdots & t_{m+1} \end{pmatrix} f_v(t_1, \dots, t_{m+1}, t). \tag{6.3}$$

Writing (6.2) with t_{m+1} replaced by t, then with t_{m+1} replaced by t', and subtracting yields

$$\begin{aligned} N^* &\begin{pmatrix} 0 & 1 & \cdots & m \\ t & t_1 & \cdots & t_m \end{pmatrix} [f_v(t_1, \dots, t_m, t) - f_v(t_1, \dots, t_m, t')] \\ &= N^* \begin{pmatrix} 0 & 1 & \cdots & m & m+1 \\ t & t_1 & \cdots & t_m & t' \end{pmatrix} f_v(t_1, \dots, t_m, t', t), \end{aligned} \tag{6.4}$$

an expression which cannot, in general, be simplified (see (1.3)).

It can be shown that

$$f_v(t_1, \dots, t_{m+1}) = \frac{F^* \begin{pmatrix} 0 & \cdots & m-1 & f \\ t_1 & \cdots & t_m & t_{m+1} \end{pmatrix}}{V^* \begin{pmatrix} 0 & \cdots & m-1 & m \\ t_1 & \cdots & t_m & t_{m+1} \end{pmatrix}}, \tag{6.5}$$

where, on the right, the numerator is similar to the denominator, with the row

$$\{v_m(t_j)\}_{j=1}^{m+1}$$

replaced by

$$\{f(t_j)\}_{j=1}^{m+1}.$$

To verify (6.5), one first shows that

$$R(t) = \frac{F^* \begin{pmatrix} 0 & \cdots & m & f \\ t_1 & \cdots & t_{m+1} & t \end{pmatrix}}{V^* \begin{pmatrix} 0 & \cdots & m \\ t_1 & \cdots & t_{m+1} \end{pmatrix}}, \tag{6.6}$$

where the numerator is defined as $v(t)$ of (5.4) with $k = m + 1$, except that for each j, $v_{m+1}(t_j)$ is replaced by $f(t_j)$. The coefficient of $v_m(t)$ in (6.6) is the negative of (6.5).

Tchebycheffian spline functions, or T-splines, are defined as were spline functions in Section 1 above, with a slight modification, namely:

$$\left.\begin{array}{l} s(x) \text{ is a } T\text{-spline of order } m \text{ or degree } m-1, \text{ if} \\ \text{(a)}\quad s(x) \text{ is a } u\text{-polynomial of degree } m-1 \text{ or less on} \\ \qquad (x_j, x_{j+1}] \text{ whenever } x_j < x_{j+1}\text{; and} \\ \text{(b)}\quad s(x) \in C^{m-i}(x_j, x_{j+i}) \text{ whenever } x_j < x_{j+i}. \end{array}\right\} \tag{6.7}$$

The reason for defining T-splines in terms of u-polynomials instead of v-polynomials will become clear after Theorem 7 below is proved.

Tchebycheffian B-splines $M_j(x)$ are defined as in Section 1, with (1.7) replaced by

$$M(x; t) = N^* \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ t & x & \cdots & x \end{pmatrix}_+ = N^* \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ t & x & \cdots & x \end{pmatrix} \quad \text{if } t \geqslant x$$
$$= 0 \quad \text{if } t < x \tag{6.8}$$

and (1.6) replaced by

$$M_j(x) = M_v(x; x_j, \ldots, x_{j+m}). \tag{6.9}$$

An early description of T-splines was given by Greville [7]. See also Ahlberg, Nilson, and Walsh [1] and Karlin and Ziegler [13].

It is not obvious that the functions $M_j(x)$ form a basis on (α, β) for the T-splines. Nor is it even apparent that the $M_j(x)$ are T-splines.

We could bypass these facts, since we now have all the machinery necessary to prove an analog of Theorem 1 and to define T-spline approximation analogous to the variation-diminishing spline approximation given by (3.3).

However, these facts do have independent interest. Moreover, their proof will place us on somewhat firmer ground. Therefore, the next section is devoted to an indication of their proofs.

7. *The $M_j(x)$ Form a Basis for the T-splines*

This section is an extension of much of Part I of Curry and Schoenberg [5]. Gaps, of which there are many, in the discussion can be filled in by looking at the corresponding places in [5].

As mentioned in Section 6 above, the facts developed in this section are not required for the definition of T-spline approximation.

Following [5], pp. 80–81, we assume that the knots $\{x_j\}$ satisfying (1.4) are located at the distinct points

$$\cdots < y_{-1} < y_0 < y_1 < \cdots \qquad (y_k \to \pm\infty \text{ as } k \to \pm\infty)$$

where for each i, y_i is a knot of multiplicity α_i ($\alpha_i \leqslant m$) with

$$y_0 = x_0 \qquad \text{and} \qquad y_1 = x_1 .$$

Then (6.7b) in the definition of T-splines becomes

$$s(x) \in C^{m-1-\alpha_i}(y_{i-1}, y_{i+1}) \qquad (-\infty < i < +\infty). \tag{6.7b$'$}$$

LEMMA 1. *Suppose that*

$$\sum_{i=1}^{i=N} \alpha_i = m + 1.$$

Then the Tchebycheffian B-spline

$$M_1(x) = M_v(x; x_1, \ldots, x_{m+1}) = M_v(x; \overbrace{y_1, \ldots, y_1}^{\alpha_1}, \ldots, \overbrace{y_N, \ldots, y_N}^{\alpha_N})$$

is a Tchebycheffian spline function. Moreover, $M_1(x)$ is precisely of the continuity class $C^{m-1-\alpha_i}$ in the neighborhood of the point y_i, for $1 \leqslant i \leqslant N$.

Proof (See also Lemma 1 of [5]).
From (6.5) and (6.9) we deduce that

$$M_1(x) = \sum_{i=1}^{N} \sum_{j=1}^{\alpha_i} a_j^i f_{ij}(x),$$

where

$$f_{ij}(x) = \left(\frac{\partial}{\partial t}\right)^{j-1} M(x; t) \Big|_{t=y_i}$$

In view of (6.8) and (5.8), either $y_i < x$ and

$$f_{ij}(x) = 0,$$

or $x \leqslant y_i$ and

$$f_{ij}(x) = \left(\frac{\partial}{\partial t}\right)^{j-1} w_m(t) \int_x^t w_{m-1}(\xi_1) \cdots \int_x^{\xi_{m-2}} w_1(\xi_{m-1})\, \delta\xi_{m-1} \cdots d\xi_1 \Big|_{t=y_i}. \tag{7.1}$$

Thus, $f_{ij}(x)$ consists of linear combinations of integrals of the form

$$\int_x^{y_i} w_{m-k}(\xi_k) \cdots \int_x^{\xi_{m-2}} w_1(\xi_{m-1})\, d\xi_{m-1} \cdots d\xi_k \qquad (0 < k \leqslant j). \tag{7.2}$$

By interchanging the order of integration, (7.2) becomes

$$\pm \int_{y_i}^{x} w_1(\xi_{m-1}) \cdots \int_{y_i}^{\xi_{k+1}} w_{m-k}(\xi_k)\, d\xi_k \cdots d\xi_{m-1}, \tag{7.3}$$

which is in the null space of L as given by (5.6) and is, hence, a u-polynomial.

Thus, on each interval $(y_i, y_{i+1}]$, $M_1(x)$ is a u-polynomial of degree at most $m - 1$, verifying (6.7a).

To verify (6.7b′), we use (6.8) again, to conclude that

$$\left(\frac{d}{dx}\right)^p M_1(y_i-) - \left(\frac{d}{dx}\right)^p M_1(y_i+)$$
$$= \sum_{j=1}^{\alpha_i} a_j{}^i \left(\frac{d}{dx}\right)^p f_{ij}(y_i-) \qquad (0 \leqslant p \leqslant m - \alpha_i). \tag{7.4}$$

From (7.1) and (7.3), $(d/dx)^p f_{ij}(x)$ for $x < y_i$ is a linear combination of terms of the form

$$\left[\left(\frac{d}{dx}\right)^{p-r} w_1(x) \cdots w_r(x)\right] \int_{y_i}^{x} w_{r+1}(\xi_{m-1-r})$$
$$\cdots \int_{y_i}^{\xi_{k+1}} w_{m-k}(\xi_k)\, d\xi_k \cdots d\xi_{m-1-r} \qquad (0 \leqslant r \leqslant p,\ 0 < k \leqslant j).$$

Evaluation of these terms at $x = y_i$ gives zero, unless $r + k = m$. Since

$$r + k \leqslant p + j < (m - \alpha_i) + \alpha_i = m,$$

the condition (6.7b′) in the definition of T-splines is verified.

Now (7.4) is also valid for $p = m - \alpha_i$. But the above analysis shows us that the terms on the right are zero except for $j = \alpha_i$, in which case $k = j = \alpha_i$ and $r = p = m - \alpha_i$. Thus, the jump in $(d/dx)^{m-\alpha_i} M_1(x)$ at $x = y_i$ is

$$\pm a^i_{\alpha_i} w_1(y_i) \cdots w_m(y_i),$$

which is nonzero, since, from (6.5) and (6.9),

$$a^i_{\alpha_i} = (-1)^{\alpha_{i+1}+\cdots+\alpha_N} \frac{V^*\begin{pmatrix} 0 & \cdots & m-1 \\ y_1 & \cdots & y_N \end{pmatrix}}{V^*\begin{pmatrix} 0 & \cdots & m \\ y_1 & \cdots & y_N \end{pmatrix}} \neq 0, \tag{7.5}$$

where the multiplicity of y_i in the numerator is $\alpha_i - 1$ instead of α_i.

This observation completes the proof of Lemma 1.

Clearly, since $M_j(x)$ involves exactly $m + 1$ knots, we can relax the supposition that

$$\sum \alpha_i = m + 1$$

and prove a corresponding lemma, regarding each $M_j(x)$, having relaxed suppositions at the knots x_j and x_{j+m}.

THEOREM 5. *Every T-spline $s(x)$ can be represented uniquely in the form*

$$s(x) = \sum_{-\infty}^{+\infty} c_j M_j(x) \qquad (c_j \text{ real}).$$

If $s(x) = 0$ outside of (y_1, y_N), then

$$s(x) = \sum_{1}^{r-m} c_j M_j(x)$$

where

$$r = \sum_{1}^{N} \alpha_i .$$

If $r \leqslant m$, $s(x) \equiv 0$.

This theorem is the analog of Theorem 5 in [5]. The proof proceeds in a straightforward manner with changes similar to those made in proving Lemma 1.

Setting $i = N$ in (7.5), gives $a_{\alpha_N}^N > 0$. This, and an interchange of the order of integration in (5.8), yields:

THEOREM 6. *Let $x_j < x_{j+1} < \cdots < x_{j+m}$. Then*

$$D_k \cdots D_2 D_1 M_j(x) \qquad (0 < k < m - 1)$$

has exactly k distinct simple zeros in the interval (x_j, x_{j+m}) and

$$M_j(x) > 0 \qquad \text{in } (x_j, x_{j+m}). \tag{7.6}$$

The operators D_i were defined in (5.6) above. The only necessary change in the proof given in [5], pp. 74–75 is to note that, in the intervals (x_i, x_{i+1}),

$$D_{m-2} \cdots D_2 D_1 M_j(x) = a_i + b_i \int_0^x w_{m-1}(x)\, dx,$$

a function which behaves like a straight line; i.e., it can have at most one zero.

The conclusion (7.6) is also valid for multiple knots. Clearly, from (6.1), (6.8), and (6.9) it follows that

$$M_j(x) = 0 \qquad \text{outside of } (x_j, x_{j+m}]. \tag{7.7}$$

It is also true that $\int_{-\infty}^{+\infty} M_j(x)\, dx = 1$. (See Radon [20], and [5], p. 74.)

8. *The B-Spline Representation of u-Polynomials*

We now extend Section 2 above.

THEOREM 7. *Let p, q be integers such that*

$$x_{p+m-1} < x_{q+1} .$$

Then the relation

$$N^*\begin{pmatrix}0 & 1 & \cdots & m-1\\ t & x & \cdots & x\end{pmatrix} = \sum_{j=p}^{j=q} a_j N^*\begin{pmatrix}0 & 1 & \cdots & m-1\\ t & x_{j+1} & \cdots & x_{j+m-1}\end{pmatrix} M_j(x), \tag{8.1}$$

is valid for all real t and for $x_{p+m-1} < x \leqslant x_{q+1}$, where the a_j are positive constants given by

$$a_j = \frac{V^*\begin{pmatrix}0 & \cdots & m\\ x_j & \cdots & x_{j+m}\end{pmatrix} V^*\begin{pmatrix}0 & \cdots & m-2\\ x_{j+1} & \cdots & x_{j+m-1}\end{pmatrix}}{V^*\begin{pmatrix}0 & \cdots & m-1\\ x_{j+1} & \cdots & x_{j+m}\end{pmatrix} V^*\begin{pmatrix}0 & \cdots & m-1\\ x_j & \cdots & x_{j+m-1}\end{pmatrix}}. \tag{8.2}$$

Proof. The proof of Theorem 1, using (6.3) and (6.4) in place of (1.2) and (1.3), extends readily up to the point of substituting (2.5) and (2.6) into (2.4), which here gives

$$N^*\begin{pmatrix}0 & 1 & \cdots & m-1\\ 1 & x & \cdots & x\end{pmatrix} = \sum_{j=p}^{j=q} \left[N^*\begin{pmatrix}0 & 1 & \cdots & m\\ t & x_j & \cdots & x_{j+m-1}\end{pmatrix}\right.$$
$$\left. - N^*\begin{pmatrix}0 & 1 & \cdots & m\\ t & x_{j+1} & \cdots & x_{j+m}\end{pmatrix}\right] M_j(x).$$

Except for the explicit form of a_j, it is clear that

$$N^*\begin{pmatrix}0 & 1 & \cdots & m\\ t & x_j & \cdots & x_{j+m-1}\end{pmatrix} - N^*\begin{pmatrix}0 & 1 & \cdots & m\\ t & x_{j+1} & \cdots & x_{j+m}\end{pmatrix}$$
$$= a_j N^*\begin{pmatrix}0 & 1 & \cdots & m-1\\ t & x_{j+1} & \cdots & x_{j+m-1}\end{pmatrix}, \tag{8.3}$$

since each of the terms on the left vanishes at $x_{j+1}, \ldots, x_{j+m-1}$ and their difference is a v-polynomial of degree at most $m-1$.

To obtain a_j explicitly, we let t tend to x_{j+m} from above. Then (8.3) yields

$$a_j = \frac{N^*\begin{pmatrix}0 & 1 & \cdots & m\\ x_{j+m}+ & x_j & \cdots & x_{j+m-1}\end{pmatrix}}{N^*\begin{pmatrix}0 & 1 & \cdots & m-1\\ x_{j+m}+ & x_{j+1} & \cdots & x_{j+m-1}\end{pmatrix}}.$$

Evaluating the right-hand member by l'Hospital's rule and then using (5.5) gives (8.2) after an appropriate number of column interchanges, completing the proof of Theorem 7.

If we let $w_i(x) = i\ (1 \leqslant i \leqslant m)$, the constant a_j reduces to

$$a_j = \frac{x_{j+m} - x_j}{m},$$

as it should. See also (2.1) and (5.7) above.

Returning now to the more general case and letting

$$N_j(x) = a_j M_j(x), \tag{8.4}$$

with a_j given by (8.2), (8.1) becomes

$$\begin{aligned} &v_{m-1}(t) - u_1(x)\, v_{m-2}(t) + u_2(x)\, v_{m-3}(t) - \cdots \\ &\quad = \sum_p^q \left[v_{m-1}(t) - \eta_j v_{m-2}(t) + \eta_j^{(2)} v_{m-3}(t) - \cdots\right] N_j(x), \end{aligned} \tag{8.5}$$

where we have utilized (5.7) on the left. The constants η_j and $\eta_j^{(2)}$ are given by

$$\eta_j = \frac{V^*\begin{pmatrix} 0 & \cdots & m-3 & m-1 \\ x_{j+1} & \cdots & x_{j+m-2} & x_{j+m-1} \end{pmatrix}}{V^*\begin{pmatrix} 0 & \cdots & m-2 \\ x_{j+1} & \cdots & x_{j+m-1} \end{pmatrix}}, \tag{8.6}$$

and

$$\eta_j^{(2)} = \frac{V^*\begin{pmatrix} 0 & \cdots & m-4 & m-2 & m-1 \\ x_{j+1} & \cdots & x_{j+m-3} & x_{j+m-2} & x_{j+m-1} \end{pmatrix}}{V^*\begin{pmatrix} 0 & \cdots & m-2 \\ x_{j+1} & \cdots & x_{j+m-1} \end{pmatrix}}.$$

Thus, we have the following corollary to Theorem 7, extending the corollary to Theorem 1.

COROLLARY. *Let $n \geqslant 1$, $x_0 = a$, $x_n = b$ and $m \geqslant 3$. Then, for $x \in [a, b]$,*

$$1 = \sum_{1-m}^{n-1} N_j(x), \tag{8.7}$$

$$u_1(x) = \sum_{1-m}^{n-1} \eta_j N_j(x), \tag{8.8}$$

and

$$u_2(x) = \sum_{1-m}^{n-1} \eta_j^{(2)} N_j(x), \tag{8.9}$$

provided that $N_{1-m}(a)$ is replaced by $N_{1-m}(a+)$ whenever $x_{-m+1} = a$.

9. *T-spline Approximation on* $[a, b]$

We assert that η_j is a nondecreasing sequence. Indeed, from (8.1) and (8.5),

$$N^*\begin{pmatrix}0 & 1 & \cdots & m-1\\ t & x_{j+1} & \cdots & x_{j+m-1}\end{pmatrix} - N^*\begin{pmatrix}0 & 1 & \cdots & m-1\\ t & x_{j+2} & \cdots & x_{j+m}\end{pmatrix}$$
$$= (\eta_{j+1} - \eta_j)\, N^*\begin{pmatrix}0 & 1 & \cdots & m-2\\ t & x_{j+2} & \cdots & x_{j+m-1}\end{pmatrix}.$$

See also (8.3) above. Letting t tend to x_{j+m} from above and using l'Hospital's rule determines $\eta_{j+1} - \eta_j$ explicitly as a positive expression similar to (8.2), unless $x_{j+1} = x_{j+m}$, in which case $\eta_{j+1} = \eta_j$. (This case will be excluded by (9.1) below.)

Let $n > 0$, $k > 0$ be integers and

$$\Delta = \{x_i\}_0^n,$$

a set of reals satisfying

$$a = x_0 < x_1 \leqslant \cdots \leqslant x_{n-1} < x_n = b$$

and

$$x_{i-k} < x_i \qquad (k < i < n). \tag{9.1}$$

We extend the set by letting

$$x_{-k} = x_{-k+1} = \cdots = x_{-1} = a,$$
$$x_{n+1} = \cdots = x_{n+k} = b.$$

We then let

$$N_j(x) \qquad (-k \leqslant j \leqslant n-1),$$

be defined by (8.4), with $k = m - 1$. We also define

$$\xi_j = u_1^{-1}(\eta_j) \qquad (-k \leqslant j \leqslant n-1), \tag{9.2}$$

with η_j given by (8.6) (see also Section 3 above).

Then

$$a = \xi_{-k} < \xi_{-k+1} < \cdots < \xi_{n-1} = b,$$

$$1 = \sum_{-k}^{n-1} N_j(x) \qquad (a \leqslant x \leqslant b), \tag{9.3}$$

and

$$u_1(x) = \sum_{-k}^{n-1} u_1(\xi_j)\, N_j(x) \qquad (a \leqslant x \leqslant b).$$

These relations are clear, except, perhaps, the assertions $\xi_{-k} = a$ and $\xi_{n-1} = b$.

The first of these is equivalent to

$$\eta_{-k} = \frac{V^*\begin{pmatrix} 0 & \cdots & m-3 & m-1 \\ a & \cdots & a & a \end{pmatrix}}{V^*\begin{pmatrix} 0 & \cdots & m-2 \\ a & \cdots & a \end{pmatrix}} = u_1(a), \tag{9.4}$$

which is easily shown by direct expansion of the determinants.

We now define *T-spline approximation.*

To each $f(x)$ defined on $[a, b]$ associate the T-spline

$$Tf(x) = T_\Delta{}^k f(x) = \sum_{j=-k}^{n-1} f(\xi_j)\, N_j(x),$$

which is to be regarded as an approximation to $f(x)$ on $[a, b]$.

This approximation method, of course, extends Schoenberg's variation diminishing spline approximation method. It has the properties:

$$Tf(x) \in C[a, b],$$

$$Tf(a) = f(a) \qquad \text{and} \qquad Tf(b) = f(b).$$

If the inequalities of (9.1) are all strict, then

$$Tf(x) \in C^{k-1}[a, b].$$

The approximation reproduces "linear" functions

$$l(x) = cu_1(x) + d.$$

Letting $w_1(x) = 1$ gives $u_1(x) = x$. This restriction amounts to the change of variable $z = u_1(x)$ and is, hence, no real restriction.

It is probable that $Tf(x)$ has the variation-diminishing property in this case. Much of Schoenberg's proof in [23] may be extended with no difficulty.

In view of (7.6) and (7.7) above, we have that

If $f(x) \geqslant 0$ on $[a, b]$, then $Tf(x) \geqslant 0$ on $[a, b]$,

which yields the following theorem.

THEOREM 8. *A necessary and sufficient condition that*

$$\lim Tf(x) = f(x), \qquad \textit{uniformly in } [a, b], \tag{9.5}$$

for every function $f(x) \in C[a, b]$, *is that* (9.5) *hold for the particular function*

$$f(x) = u_2(x),$$

or, equivalently, that

$$E(x) = Tu_2(x) - u_2(x) \tag{9.6}$$

converges uniformly to 0.

With this theorem, we have extended Part I, except for the variation-diminishing property and Section 4. We shall now prove a result analogous to (4.1), which will yield weaker versions of Theorems 3 and 4.

LEMMA. *Let* $A_i = \max_{[a,b]} w_i(x)$ $(i = 1, 2)$ *and* $B = A_1 A_2(| a | + | b |)$. *Then*

$$0 \leqslant | E(x)| \leqslant Bk \| \Delta \|. \tag{9.7}$$

Proof. From (9.6) and (5.1),

$$E(x) = \sum_{-k}^{n-1} \int_x^{\xi_j} w_1(\tau_1) \int_0^{\tau_1} w_2(\tau_2)\, d\tau_2\, d\tau_1\, N_j(x),$$

and, hence,

$$\begin{aligned} | E(x)| &\leqslant \tfrac{1}{2} A_1 A_2 \sum_{-k}^{n-1} | \xi_j^2 - x^2 | N_j(x) \\ &\leqslant \tfrac{1}{2} A_1 A_2 \sum_{J_x} | \xi_j^2 - x^2 | N_j(x), \end{aligned}$$

where J_x is the set of j's for which $N_j(x) \neq 0$. Using (8.7), we have

$$\begin{aligned} | E(x)| &\leqslant \tfrac{1}{2} A_1 A_2 \max_{J_x} | \xi_j^2 - x^2 | \\ &\leqslant \tfrac{1}{2} A_1 A_2 \max | \xi_j + x | \max_{J_x} | \xi_j - x | \\ &\leqslant A_1 A_2(| a | + | b |) \max(x_{j+k} - x_j), \end{aligned}$$

since

$$x_{j+1} \leqslant \xi_j \leqslant x_{j+k}\,.$$

Proof of this last inequality is similar to the proof of the first of the relations (9.3). Continuing, we have

$$\begin{aligned} | E(x)| &\leqslant A_1 A_2(| a | + | b |)\, k \| \Delta \| \\ &= Bk \| \Delta \|, \end{aligned}$$

which completes the proof.

Theorem 8 and the last Lemma yield weaker forms of Theorems 3 and 4. Theorem 3 becomes:

A sufficient condition that

$$\lim Tf(x) = f(x), \qquad \textit{uniformly in } [a, b], \tag{9.8}$$

for every $f(x) \in C[a, b]$, *is that*

$$\lim k \, \| \Delta \| = 0. \tag{9.9}$$

A similar change occurs in Theorem 4.

Undoubtedly stronger statements can be made. However, it may be necessary to require that the $u_i(x)$ satisfy some condition similar to that encountered in Müntz's theorem.

We close this section, and Part II, by observing that the requirement that

$$w_i(x) \in C^{m+1}(-\infty, +\infty)$$

is not needed, provided that the continuity requirements for T-splines, $M_j(x)$ and $Tf(x)$ be correspondingly relaxed.

Part III: Some Properties of Spline Approximation

10. *Derivatives of Variation-Diminishing Spline Approximations*

It is known (see Davis [6], p. 113) that for functions $f(x)$ contained in $C^p[0, 1]$, the pth derivatives of the corresponding Bernstein polynomials converge uniformly to the pth derivative of $f(x)$ on [0, 1].

This fact does *not* hold for variation diminishing spline approximations in general, except for $p = 1$.

Indeed, for the special case of equidistant knots,

$$n > 1 \qquad \text{and} \qquad x_i = \frac{i}{n} \qquad (0 < i < n),$$

the rth derivative $(1 < r \leqslant p)$ of the spline approximation of degree k to $f(x)$ will converge to $f^{(r)}(x)$ as

$$\frac{\| \Delta \|}{k} \to 0 \qquad \text{or, equivalently,} \qquad k + n \to \infty$$

if and only if

$$0 < x < 1.$$

The convergence is uniform on compact subintervals of (0, 1).

In this section, we shall concentrate on the first and second derivatives only.

The next three paragraphs serve to introduce the notation which we shall follow in this section.

By D and D^2 we shall mean the first- and second-derivative operators. By

$$S_\Delta^k f(x) = S_\Delta f(x) = \sum_{-k}^{n-1} f(\xi_j)\, N_j(x),$$

we shall, as usual, mean the variation-diminishing spline approximation to $f(x)$ of degree k, over the knots of

$$\Delta = \{x_i\}_{-k}^{n+k},$$

where

$$\begin{aligned} x_{-k} = x_{-k+1} = \cdots = x_0 = 0 < x_1\,, \\ x_{n-1} < x_n = \cdots = x_{n+k} = 1, \\ x_{i-1} \leqslant x_i \qquad (1 < i < n), \end{aligned} \tag{10.1}$$

and

$$x_{i-k} < x_i \qquad (k < i < n).$$

In order that the operators S_{Δ_-} and $S_{\Delta_=}$ below have strictly increasing sequences of nodes, we shall add the requirement that whenever $k > 2$,

$$x_{i-k} < x_{i-2} \qquad (k < i < n + 2).$$

This requirement is really not necessary, but it avoids an awkward segmentation of the interval [0, 1] at "bad spots" which would otherwise occur.

By

$$S_{\Delta_-}^{k-1} f(x) = S_{\Delta_-} f(x) = \sum_{-(k-1)}^{n-1} f(\xi_j^-)\, N_j^-(x),$$

we shall mean the variation-diminishing spline approximation of degree $k - 1$ over the knots of Δ_-, where

$$\Delta_- = \Delta - \{x_{-k}\,, x_{n+k}\} = \{x_i\}_{-(k-1)}^{n+(k-1)}.$$

Here, ξ_j^- and $N_j^-(x)$ denote the nodes and fundamental functions corresponding to Δ_-.

Similarly, we denote

$$\Delta_= = \{x_i\}_{-(k-2)}^{n+(k-2)}$$

and let

$$S_{\Delta_=}^{k-2} f(x) = S_{\Delta_=} f(x) = \sum_{-(k-2)}^{n-1} f(\xi_j^=)\, N_j^=(x),$$

denote the variation-diminishing spline approximation of degree $k-2$ over the knots of $\Delta_=$.

The following lemma is the key result of this section:

LEMMA 1. *Let $f(x) \in C'[0, 1]$ and let $k > 1$. Then*

$$DS_\Delta f(x) = \sum_{1-k}^{n-1} \frac{f(\xi_j) - f(\xi_{j-1})}{\xi_j - \xi_{j-1}} N_j^-(x), \tag{10.2}$$

and there is a smooth function $r(x)$ such that

$$\sup_{[0,1]} |r(x)| \leqslant \omega(\max_j (\xi_j - \xi_{j-1}); Df) \tag{10.3}$$

and

$$DS_\Delta f(x) = S_{\Delta\_}[Df(x) + r(x)], \tag{10.4}$$

where $\omega(\delta; g)$ is the modulus of continuity of $g(x)$ on [0, 1].

Proof. Since

$$DS_\Delta f(x) = \sum_{-k}^{n-1} f(\xi_j)\, DN_j(x), \tag{10.5}$$

we examine the functions $DN_j(x)$. At each of the knots $x_j, x_{j+1}, \ldots, x_{j+k+1}$ the order of continuity of $DN_j(x)$ is precisely one less than that of $N_j(x)$. Also, $DN_j(x)$ is of degree $k-1$. Thus, $DN_j(x)$ is a spline function of degree $k-1$ over the knots of Δ_-.

Since the support of $DN_j(x)$ is $[x_j, x_{j+k+1}]$, $DN_j(x)$ is a uniquely determined linear combination of those $N_i^-(x)$ with support in $[x_j, x_{j+k+1}]$. See Theorem 5 in [5]. By considering the orders of continuity at x_j and x_{j+k+1}, we see that, in fact,

$$DN_j(x) = \alpha_j N_j^-(x) - \beta_{j+1} N_{j+1}^-(x) \qquad (-k < j < n-1). \tag{10.6}$$

Before evaluating α_j and β_{j+1}, we shall consider $DN_{-k}(x)$ and $DN_{n-1}(x)$, both of which are not covered by (10.6).

From the explicit form of $N_{-k}(x)$ and $N_{n-1}(x)$ on [0, 1], we arrive at

$$DN_{-k}(x) = D\left(\frac{x_1 - x}{x_1}\right)_+^k = \frac{-k}{x_1} N_{1-k}^-(x) = \frac{-N_{1-k}^-(x)}{\xi_{1-k} - \xi_{-k}}, \tag{10.7}$$

and

$$DN_{n-1}(x) = D\left(\frac{x - x_{n-1}}{1 - x_{n-1}}\right)_+^k = \frac{N_{n-1}^-(x)}{\xi_{n-1} - \xi_{n-2}}. \tag{10.8}$$

To determine the constants in (10.6), it is convenient to use the relations

$$\sum_{-k}^{n-1} N_j(x) = 1 \qquad (0 \leqslant x \leqslant 1) \tag{10.9}$$

and

$$\sum_{-k}^{n-1} \xi_j N_j(x) = x \qquad (0 \leqslant x \leqslant 1). \tag{10.10}$$

From (10.9) we have

$$\sum_{-k}^{n-1} DN_j(x) = 0 \qquad (0 \leqslant x \leqslant 1). \tag{10.11}$$

Substituting (10.6), (10.7), and (10.8) into (10.11) and reindexing gives

$$N_{1-k}^{-}(x) \left[\alpha_{1-k} - \frac{1}{\xi_{1-k} - \xi_{-k}}\right] + \sum_{2-k}^{n-2} (\alpha_j - \beta_j)\, N_j^{-}(x)$$
$$+ N_{n-1}^{-}(x) \left[\frac{1}{\xi_{n-1} - \xi_{n-2}} - \beta_{n-1}\right] = 0.$$

By uniqueness of representation, each coefficient is zero, so that

$$\begin{aligned} \alpha_{1-k} &= \frac{1}{\xi_{1-k} - \xi_{-k}}, \\ \alpha_j &= \beta_j \qquad (1 - k < j < n - 1), \\ \beta_{n-1} &= \frac{1}{\xi_{n-1} - \xi_{n-2}}. \end{aligned} \tag{10.12}$$

Differentiation of (10.10) yields

$$\sum_{-k}^{n-1} \xi_j\, DN_j(x) = 1 \qquad (0 \leqslant x \leqslant 1).$$

Substituting and reindexing gives

$$N_{1-k}^{-}(x) + \sum_{2-k}^{n-2} \alpha_j(\xi_j - \xi_{j-1})\, N_j^{-}(x) + N_{n-1}^{-}(x) = 1,$$

from which uniqueness of representation gives

$$\alpha_j = \frac{1}{\xi_j - \xi_{j-1}} \qquad (-k + 1 < j < n - 1). \tag{10.13}$$

Substitution of these results into (10.5) and reindexing yields (10.2). Thus, we have proved the first assertion of the lemma.

The second assertion (regarding $r(x)$) now follows readily, since

$$\xi_{j-1} \leqslant \xi_j^- \leqslant \xi_j \qquad (-k < j < n), \tag{10.14}$$

and, by the mean-value theorem, for each j there is an η_j such that

$$\xi_{j-1} < \eta_j < \xi_j\,, \tag{10.15}$$

and

$$\frac{f(\xi_j) - f(\xi_{j-1})}{\xi_j - \xi_{j-1}} = Df(\eta_j).$$

Thus,

$$DS_\Delta f(x) = \sum_{1-k}^{n-1} [Df(\eta_j)]\, N_j^-(x) = \sum_{1-k}^{n-1} [Df(\xi_j^-) + r(\xi_j^-)]\, N_j^-(x)$$

where

$$r(\xi_j^-) = Df(\eta_j) - Df(\xi_j^-).$$

Now

$$\max |\, r(\xi_j^-)| \leqslant \omega(\max |\, \eta_j - \xi_j^- |; Df) \leqslant \omega(\max(\xi_j - \xi_{j-1}); Df),$$

since (10.14) and (10.15) imply that

$$|\, \eta_j - \xi_j^- \,| < \xi_j - \xi_{j-1}\,.$$

Clearly, $r(x)$ may be defined on all of $[0, 1]$ as a smooth function with extrema only at the nodes ξ_j^-.

This completes the proof of Lemma 1.

The following theorem extends a well-known property of Bernstein polynomials (see Natanson [17], p. 179, or Davis [6], p. 113). The surprising fact is that it is not true if D is replaced by D^2.

THEOREM 9. *Let* $f(x) \in C'[0, 1]$. *Let*

$$\frac{\|\Delta\|}{k} \to 0 \tag{10.16}$$

and

$$\liminf k > 1. \tag{10.17}$$

Then

$$S_\Delta f(x) \to f(x) \qquad \textit{uniformly on } [0, 1] \tag{10.18}$$

and

$$DS_\Delta f(x) \to Df(x) \qquad \textit{uniformly on } [0, 1]. \tag{10.19}$$

Proof. That (10.16) implies (10.18) follows from Theorem 3 of Section 4 in Part I above.

Now, (10.17) implies that eventually

$$\frac{\| \Delta_- \|}{k-1} = \frac{\| \Delta \|}{k-1} \leqslant 2 \frac{\| \Delta \|}{k}.$$

Thus, in view of (10.16),

$$\frac{\| \Delta_- \|}{k-1} \to 0,$$

which implies (again by Theorem 3) that

$$S_{\Delta_-} Df(x) \to Df(x) \qquad \text{uniformly on } [0, 1].$$

But (10.4), the variation-diminishing property, and (10.3), respectively, imply that

$$\begin{aligned} | DS_\Delta f(x) - S_{\Delta_-} Df(x) | &= | S_{\Delta_-} r(x) | \\ &\leqslant \sup_{[0,1]} | r(x) | \leqslant \omega(\max(\xi_j - \xi_{j-1}); Df). \end{aligned} \tag{10.20}$$

Since $f(x) \in C'[0, 1]$ and $\max(\xi_j - \xi_{j-1}) \to 0$ (see Bohman [4], pp. 43–45), the right member of (10.20) approaches zero.

Observing that this approach is independent of x completes the proof of (10.19) and, hence, of the theorem.

Whenever $k > 1$ and $f(x) \in C^2[0, 1]$, we may apply (10.2) twice and use remainder theory to obtain the following result. (See also Gruss [8] and Popovicui [19].)

LEMMA 2. *Let $f(x) \in C^2[0, 1]$ and $k > 1$. Then*

$$\begin{aligned} D^2 S_\Delta f(x) &= \sum_{2-k}^{n-1} \frac{\dfrac{f(\xi_j) - f(\xi_{j-1})}{\xi_j - \xi_{j-1}} - \dfrac{f(\xi_{j-1}) - f(\xi_{j-2})}{\xi_{j-1} - \xi_{j-2}}}{\xi_j^- - \xi_{j-1}^-} N_j^=(x) \\ &= \sum_{2-k}^{n-1} D^2 f(\eta_j) \frac{\xi_j - \xi_{j-2}}{2(\xi_j^- - \xi_{j-1}^-)} N_j^=(x), \end{aligned} \tag{10.21}$$

where

$$\xi_{j-2} < \eta_j < \xi_j. \tag{10.22}$$

If the convergence criterion

$$\frac{\| \Delta \|}{k} \to 0 \qquad (\text{hence, } \max | \xi_j^= - \eta_j | \to 0), \tag{10.23}$$

is satisfied, the difference between $D^2f(\eta_j)$ and $D^2f(\xi_j^=)$ becomes uniformly small. Thus, the limiting behavior of $D^2S_\Delta f(x)$ relative to $D^2f(x)$ depends solely upon the behavior of the factors

$$B_j = \frac{\xi_j - \xi_{j-2}}{2(\xi_j^- - \xi_{j-1}^-)} \qquad (1 - k < j < n). \tag{10.24}$$

Unfortunately, none of our previous assumptions will impose "nice" behavior on the B_j. For example, at $x = 0$,

$$D^2S_\Delta f(0) = D^2f(\eta_{2-k}) \frac{k-1}{k} \frac{x_1 + x_2}{2x_1},$$

the term

$$\frac{k-1}{k} \frac{x_1 + x_2}{2x_1}, \tag{10.25}$$

being B_{2-k}. From (10.22),

$$0 < \eta_{2-k} < \frac{x_1 + x_2}{k},$$

so that (10.23) implies that

$$D^2f(\eta_{2-k}) \to D^2f(0).$$

But the remaining factor (10.25) may tend toward any number greater than or equal to $\frac{1}{2}$ (since $(k-1)/k \geqslant \frac{1}{2}$ and $(x_1 + x_2)/2x_1 \geqslant 1$), or may have no limit at all.

In a similar manner, since each B_j is at least $(k-1)/k$, it may be shown that for any $x \in [0, 1]$, the ratio

$$\frac{D^2S_\Delta f(x)}{D^2f(x)},$$

may have any limit greater than or equal to $\frac{1}{2}$, or may have no limit at all. The behavior is independent of the particular $f(x)$ chosen from $C^2[0, 1]$.

We do have the following:

THEOREM 10. *Let $f(x) \in C^3[0, 1]$ and $k > 2$. Then*

(A) *If $Df(x) \geqslant 0$ on $[0, 1]$, then $DS_\Delta f(x) \geqslant 0$ on $[0, 1]$.*

(B) *If $D^2f(x) \geqslant 0$ on $[0, 1]$, then $D^2S_\Delta f(x) \geqslant 0$ on $[0, 1]$.*

However,

(C) *If $D^3f(x) \geqslant 0$ on $[0, 1]$, $D^3S_\Delta f(x)$ need not be nonnegative.*

A similar (and more basic) result holds if the derivatives are replaced by divided differences (see Gruss [8] and Popoviciu [19]). In particular, convexity of order 0 or 1 is preserved, but convexity of order 2 is not (see [19] for definition of convexity or order n).

Proof of Theorem 10. Statements A, B follow from Lemmas 1, 2, respectively.

To verify C, we set $g(x) = x^2$. Then $D^3g(x) \geqslant 0$. By using Lemma 1 three times, with

$$x_i = \frac{i}{n} \qquad (0 < i < n) \qquad \text{and} \qquad k > 2,$$

we get

$$D^3S_\Delta g(x) = \sum_{3-k}^{n-1} \frac{\dfrac{\xi_j - \xi_{j-2}}{\xi_j^- - \xi_{j-1}^-} - \dfrac{\xi_{j-1} - \xi_{j-3}}{\xi_{j-1}^- - \xi_{j-2}^-}}{\xi_j^= - \xi_{j-1}^=} N_j^=(x)$$

and, in particular,

$$D^3S_\Delta g(0) = \frac{-n(k-1)(k-2)}{2k} < 0.$$

We now state a theorem about the limiting behavior of variation diminishing spline approximation over equidistant knots.

THEOREM 11. *Let $f(x) \in C^2[0, 1]$, and let*

$$x_i = \frac{i}{n} \qquad (0 < i < n),$$

be the interior knots of Δ. Let $n + k \to \infty$, $\liminf n > 1$, *and* $\liminf k > 1$. *If*

$$\lim \left(\frac{k-1}{k}\right) = R$$

exists, then

$$\lim D^2S_\Delta f(0) = \frac{3R}{2} D^2f(0),$$

$$\lim D^2S_\Delta f(1) = \frac{3R}{2} D^2f(1),$$

and

$$\lim D^2S_\Delta f(x) = D^2f(x) \qquad (0 < x < 1).$$

The convergence is uniform on compact subsets of (0, 1).

Theorem 11 follows from Lemma 2 by a proof similar to that of Theorem 1 in [16].

A separate argument is needed for $R = \frac{1}{2}$ ($\lim k = 2$).

If $R > \frac{1}{2}$, it can be shown that there are functions

$$b_{nk}(x) \qquad (n > 1,\, k > 2,\, 0 \leqslant x \leqslant 1), \tag{10.26}$$

for which (10.24) becomes

$$B_j = b_{nk}(\xi_j^{=}). \tag{10.27}$$

Then (10.21) gives

$$\lim D^2 S_\Delta f(x) = \lim_{n+k\to\infty} b_{nk}(x)\, D^2 f(x),$$

using an argument similar to the proof of (10.3) and (10.4). It can also be shown that

$$\lim_{n+k\to\infty} b_{nk}(x) = \begin{cases} \frac{3}{2}R & (x = 0, 1) \\ 1 & (0 < x < 1). \end{cases} \tag{10.28}$$

A complete proof would require specification of the $b_{nk}(x)$. Here we shall observe only that, for x's near zero,

$$b_{nk}(x) = \frac{k-1}{k} + \frac{k-1}{k}\{1 + [1 + 8n(k-2)x]^{1/2}\}^{-1}, \tag{10.29}$$

suggesting the reason for (10.28). To arrive at (10.29), one first expresses B_j as given by (10.24) as a function of j, then expresses j as a function of $\xi_j^{=}$. Combining the two, and replacing $\xi_j^{=}$ by x, yields $b_{nk}(x)$. It is necessary to segment the interval $[0, 1]$ at the points $(k-2)/2n$ and $1 - (k-2)/2n$ if $k - 2 \leqslant n$, or $n/2(k-2)$ and $1 - n/2(k-2)$ if $n \leqslant k - 2$.

11. *Spline Approximation to Convex Functions*

In this section we consider the effect of "refining" Δ, on spline approximation to convex functions. More specifically, we consider

$$\Delta = \{x_i\}_{-k}^{n+k},$$

and

$$\Delta' = \{x_i'\}_{-k'}^{n'+k'},$$

where both Δ and Δ' satisfy (10.1) above and

$$\Delta' \supset \Delta \qquad (\text{hence, } k' \geqslant k,\, n' \geqslant n),$$

and ask whether the following conjecture is valid.

CONJECTURE. If $f(x)$ is convex, then

$$S_\Delta f(x) \geqslant S_{\Delta'} f(x) \qquad (0 \leqslant x \leqslant 1).$$

This conjecture is suggested by the fact that it is valid for Bernstein polynomials; that is, when $n = n' = 1$ (see Schoenberg [21] or Davis [6], p. 115).

It is not true when $n = n' > 1$; in fact, we have

THEOREM 12. *Whenever*

$$n = n' > 1 \qquad \textit{and} \qquad k' > k,$$

there is a convex function $f(x)$ for which

$$S_\Delta f(x_1) < S_{\Delta'} f(x_1),$$

where x_1 is the first interior knot of Δ.

Proof. There is an integer j for which

$$0 < x_1 = \cdots = x_j < x_{j+1}\,.$$

Then

$$\xi_{j-k} = \frac{j}{k}\, x_1 \leqslant x_1\,,$$

and

$$\xi'_{j-k'} = \frac{j}{k'}\, x_1 < \xi_{j-k}\,.$$

Let

$$\begin{aligned} f(x) &= (\xi_{j-k} - x)(\xi_{j-k} - \xi'_{j-k'})^{-1} && (0 \leqslant x \leqslant \xi_{j-k}), \\ &= 0 && (\xi_{j-k} \leqslant x \leqslant 1). \end{aligned}$$

Then $f(x)$ is convex,

$$f(\xi_i) = 0 \qquad \text{for} \quad i \geqslant j - k,$$

and

$$N_i(x_1) = 0 \qquad \text{for} \quad i < j - k,$$

so that

$$f(\xi_i)\, N_i(x_1) = 0 \qquad \text{for all } i.$$

Also,

$$f(\xi'_{j-k'})\, N'_{j-k'}(x_1) = N'_{j-k'}(x_1) > 0,$$

since the support of $N'_{j-k'}(x)$ is $[0, x_{j+1}]$.

Thus,

$$S_{\Delta}f(x_1) = 0 < N'_{j-k'}(x_1) \leqslant S_{\Delta'}f(x_1),$$

contradicting the conjecture. The latter inequality follows from the variation-diminishing property.

When $k = k'$, the conjecture is valid. In a manner similar to the proof of (10.2) above, one can prove the following lemma and, hence, the theorem with which we conclude this section.

LEMMA. *Let $f(x)$ be a convex function and let*

$$\Delta' \supset \Delta \qquad \textit{with} \quad n' = n + 1 \quad \textit{and} \quad k' = k.$$

Then

$$S_{\Delta}f(x) \geqslant S_{\Delta'}f(x) \qquad (0 \leqslant x \leqslant 1).$$

THEOREM 13. *Let $f(x)$ be a convex function and let*

$$\Delta' \supset \Delta \qquad \textit{with} \quad k' = k.$$

Then

$$S_{\Delta}f(x) \geqslant S_{\Delta'}f(x) \qquad (0 \leqslant x \leqslant 1).$$

Observe that Theorem 13, which is proved by applying the lemma $n'-n$ times, is encouraging, since, in practical application of spline approximation theory, one would normally want to fix k when refining Δ.

In view of the relaxation in continuity which would occur, it seems plausible to improve spline approximation to arbitrary functions by inserting knots at points where the error in the current approximation is worst.

12. *Constraints on the Nodes*

We now consider spline approximation of a given positive degree k where the nodes

$$\xi_j \qquad (-k \leqslant j \leqslant n - 1)$$

are given, but the knots

$$x_i \qquad (-k \leqslant i \leqslant n + k)$$

are not specified. We suppose that the nodes satisfy

$$0 = \xi_{-k} < \xi_{1-k} < \cdots < \xi_{n-1} = 1. \tag{12.1}$$

This is the form in which a practical problem might arise, with both ξ_j and $f(\xi_j)$ specified and a spline approximation to $f(x)$ desired.

Since

$$\xi_j = \frac{x_{j+1} + \cdots + x_{j+k}}{k} \qquad (-k \leqslant j < n), \tag{12.2}$$

there are $k - 1$ degrees of freedom on the knots

$$x_i \qquad (-k < i < n + k).$$

The choice of x_{-k} and x_{n+k} is irrelevant to the resulting approximation, provided that for each i

$$x_{-k} \leqslant x_i \leqslant x_{n+k} \,.$$

To specify the knots, we impose the condition

$$x_{1-k} = x_{2-k} = \cdots = x_{-1} = 0 \qquad \text{(hence, } x_0 = 0). \tag{12.3}$$

We shall say that a set of nodes "yields a spline approximation of degree k" if it satisfies (12.1) and if (12.2) and (12.3) imply that

$$x_{j-1} \leqslant x_j \qquad (0 < j < n + k). \tag{12.4}$$

A set of nodes which does *not* yield a spline approximation of degree $k = 2$ is the following:

$$\xi_{-2} = 0, \qquad \xi_{-1} = 0.4, \qquad \xi_0 = 0.7, \qquad \xi_1 = 1,$$

for which (12.2) and (12.3) imply

$$x_1 = 0.8, \qquad x_2 = 0.6, \qquad x_3 = 1.4,$$

in violation of (12.4).

In Theorem 14, a necessary and sufficient condition that the nodes yield a spline approximation of a fixed degree k is given. In its corollary, a condition that the nodes yield spline approximations of all degrees is given.

Theorem 15 gives a condition that a set of nodes which yields a spline approximation of degree k will also yield knots satisfying

$$x_n = x_{n+1} = \cdots = x_{n+k-1} = 1. \tag{12.5}$$

It is only in this case that the approximation is "good" on the full interval $[0, 1]$ and that we have the variation diminishing spline approximation described in Section 3 above.

In the corollaries to this theorem, the consequences of equal spacing of the nodes are demonstrated, and it is "shown" that linear interpolation ($k = 1$) is always possible.

All of these results follow from the following:

LEMMA. *For* $0 < j < n + k$,

$$x_j = k \sum_{i=1}^{(j+k-1)/k} (\xi_{j-ik} - \xi_{j-1-ik}) = k\nabla \sum_{i=1}^{(j+k-1)/k} \xi_{j-ik} \,. \tag{12.6}$$

Here

$$\nabla \eta_j = \eta_j - \eta_{j-1},$$

is the first backward difference. We shall later need

$$\nabla^2 \eta_j = \eta_j - 2\eta_{j-1} + \eta_{j-2}.$$

Proof. For convenience, let $y_j = x_j/k$. Then (12.2) implies

$$y_{j+k} = \xi_j - \xi_{j-1} + y_j \qquad (-k < j < n). \tag{12.7}$$

In view of (12.3),

$$\begin{aligned} y_1 &= \xi_{1-k} - \xi_{-k}, \\ &\cdots \\ y_k &= \xi_0 - \xi_{-1}, \end{aligned} \tag{12.8}$$

which is (12.6) for $0 < j \leqslant k$.

Combining (12.7) and (12.8) for $k < j \leqslant 2k$ yields

$$\begin{aligned} y_{k+1} &= (\xi_1 - \xi_0) + (\xi_{1-k} - \xi_{-k}), \\ &\cdots \\ y_{2k} &= (\xi_k - \xi_{k-1}) + (\xi_0 - \xi_1). \end{aligned} \tag{12.9}$$

By induction the complete lemma follows.

THEOREM 14. *Let $k > 0$. A necessary and sufficient condition that a set of nodes ξ_j yield a spline approximation of degree k is that*

$$\nabla^2 \left(\sum_{i=1}^{(j+k-1)/k} \xi_{j-ik} \right) \geqslant 0 \qquad (1 < j < n + k). \tag{12.10}$$

This spline approximation is variation diminishing on the interval $[0, x_n]$.

Proof. The expression $x_j - x_{j-1}$ is the left-hand side of (12.10). Adding knots at x_{n+k-1} to bring the multiplicity there up to k will yield the variation-diminishing spline approximation of Section 3 above, on the interval $[0, x_{n+k-1}]$. But the added $N_j(x)$ do not have support on $[0, x_n]$.

COROLLARY. *A necessary and sufficient condition that a set of numbers*

$$0 = \eta_0 < \eta_1 < \cdots < \eta_l = 1$$

yield spline approximations of all degrees is that

$$\nabla^2(\eta_j) \geqslant 0 \qquad (1 < j \leqslant l), \tag{12.11}$$

or, equivalently, that the nodes yield a spline approximation of degree $k = l$.

No upper limit on k has been assumed so far. A natural upper limit, $k \leqslant l$, arises when we require that the spline approximation yielded by a set of nodes reduces to that described in Section 3.

THEOREM 15. *Let $k > 0$ and the nodes ξ_j yield a spline approximation of degree k. Then a necessary and sufficient condition that the spline approximation be variation diminishing on all of* [0, 1] *is that*

$$k\nabla\left(\sum_{i=1}^{(n+k-1)/k} \xi_{n-ik}\right) = 1. \tag{12.12}$$

Proof. The left-hand side of (12.12) is x_n.

COROLLARY 1. *If the nodes ξ_j are evenly spaced, with*

$$\xi_j = \frac{j+k}{n+k-1} \qquad (-k \leqslant j < n), \tag{12.13}$$

then these nodes yield a spline approximation of degree k. This spline approximation is equivalent to Bernstein polynomial approximation of degree k in the intervals

$$\frac{ik-k}{n+k-1} \leqslant x \leqslant \frac{ik}{n+k-1} \qquad \left(0 < i \leqslant \frac{n+k-1}{k}\right).$$

If $(n-1)/k$ is an integer, (12.5) *is satisfied. Conversely, if $(n-1)/k$ is not an integer,* (12.5) *is not satisfied.*

Proof. Theorem 14 yields the first statement. The second statement follows, since the knots are equal in groups of k, with the possible exception of the last group (those which are outside the interval $0 \leqslant x < 1$). The last statements follow from (12.12) and (12.13).

The following not-so-astounding corollary is mentioned to verify, once again, that linear interpolation is the "best" approximation of all, this time, because no constraints on the nodes are necessary (see also the Introductions to both [16] and [22]).

COROLLARY 2. *Linear interpolation, that is, variation diminishing spline approximation with $k = 1$, is always possible.*

Proof. The statement is equivalent to saying that when $k = 1$, (12.10) and (12.12) are implied by (12.1) without additional conditions. This is easily verified.

PART IV: CONCERNING VORONOVSKAYA'S THEOREM

13. *An Analog of Voronovskaya's Theorem*

For $n = 1$, variation diminishing spline approximation reduces to Bernstein polynomial approximation. We have already seen that many properties of the latter extend. We shall now consider whether Voronovskaya's theorem (see [6] or [15]), which says that

$$\lim k[Sf(x) - f(x)] = f''(x)\, x(1 - x)/2 \qquad (n = 1) \tag{13.1}$$

for bounded functions, where $f''(x)$ exists, will similarly extend.

Recalling from Section 4 that for $g(x) = x^2$, $n = 1$, we have

$$E(x) = Sg(x) - g(x) = \frac{x(1-x)}{k},$$

we see that (13.1) can be written

$$\lim k[Sf(x) - f(x)] = f''(x)\left[\lim k\,\frac{E(x)}{2}\right].$$

Since spline approximation involves the interior knots of Δ (i.e., x_i, for $0 < i < n$) as well as the degree k, we should expect that the letter k would be replaced by an expression involving k and the knots of Δ; for example,

$$\lim h(k; \Delta)[Sf(x) - f(x)] \stackrel{?}{=} f''(x)\left[\lim h(k; \Delta)\,\frac{E(x)}{2}\right], \tag{13.2}$$

where $h(k; \Delta)$ is such that the limit on the right exists but is not trivial.

That (13.2) can be valid in more general instances is shown by the following theorem. Observe that k shows up in the denominator rather than the numerator. This theorem was stated by I. J. Schoenberg in [22]. It is not valid if $\lim k/n > 0$.

THEOREM 16. *Let the spline approximation of Section* 3 *above be defined with the interior knots of Δ given by*

$$x_i = \frac{i}{n} \qquad (0 < i < n).$$

Let $0 < x' < 1$. *If* $f(x) \in C''[0, 1]$ *and*

$$\lim \frac{k}{n} = 0,$$

then

$$\lim \frac{n^2}{k+1} [Sf(x') - f(x')] = \frac{f''(x')}{2} \lim \frac{n^2}{k+1} E(x') = \frac{f''(x')}{24}. \tag{13.3}$$

Proof. From Taylor's theorem,

$$f(\xi_j) = f(x') + f'(x')(\xi_j - x') + \frac{f''(x')}{2} (\xi_j - x')^2 + s(\xi_j)(\xi_j - x')^2, \tag{13.4}$$

where

$$\lim_{x \to x'} s(x) = 0.$$

Thus,

$$Sf(x') = f(x') + \frac{f''(x')}{2} E(x') + \sum_{-k}^{n-1} s(\xi_j)(\xi_j - x')^2 N_j(x'), \tag{13.5}$$

and hence

$$\begin{aligned} &\frac{n^2}{k+1} \left| \left[Sf(x') - f(x') - \frac{f''(x')}{2} E(x') \right] \right| \\ &\leqslant [\sup_{|x-x'|<(k+1)/2n} | s(x) |] \frac{n^2}{k+1} E(x') \\ &\leqslant \omega \left(\frac{k+1}{2n} ; s \right) \Big/ 12, \end{aligned} \tag{13.6}$$

since $N_j(x') = 0$ for j such that $| \xi_j - x' | > (k+1)/2n$, and it was shown in [16] that

$$E(x) \leqslant \frac{(k+1)}{12n^2} \qquad (k < n+2).$$

As $(k+1)/n \to 0$, the right-hand side of (13.6) tends to zero. Since it was also shown in [16] that $E(x) = (k+1)/12n^2$ whenever

$$(k+1)/n < x < 1 - (k+1)/n,$$

we have (13.3).

In the more general situation represented by (13.2), one would proceed in a similar manner, getting

$$\begin{aligned} &h(k; \Delta) \left[Sf(x) - f(x) - \frac{f''(x)}{2} E(x) \right] \\ &= h(k; \Delta) \sum_{-k}^{n-1} s(\xi_j)(\xi_j - x)^2 N_j(x). \end{aligned} \tag{13.7}$$

To "prove" that the right-hand side tends to zero, we split the sum into two parts,

$$\sum_{-k}^{n-1} = \sum_1 + \sum_2 , \tag{13.8}$$

where $\sum_1$ denotes summation over those j for which

$$| \xi_j - x | < \delta,$$

and $\sum_2$ denotes summation over those j for which

$$| \xi_j - x | \geqslant \delta.$$

The number δ is chosen so that in $\sum_1$, $s(\xi_j)$ is very small and so that

$$h(k; \Delta) \sum_2 (\xi_j - x)^2 N_j(x), \tag{13.9}$$

is very small, too. Since the remaining factors in each sum are bounded, the "proof" is complete.

The difficult part of the proof is in showing that (13.9) is very small.

14. *Two Conjectures*

We illustrate the procedure by partially proving the following conjecture.

CONJECTURE 1. *Let $f(x)$ be bounded in* $[0, 1]$. *Let*

$$l = n + k - 1,$$

and let

$$x_i = \frac{i}{n} \qquad (0 < i < n)$$

be the interior knots of Δ. If

$$\lim_{l \to \infty} \frac{k}{n} = t, \tag{14.1}$$

exists as a positive extended real number, then

$$\lim_{l \to \infty} lE(x) = e(x, t) \qquad (0 \leqslant x \leqslant 1) \tag{14.2}$$

exists, is a continuous function of x, and

$$\lim_{l \to \infty} l[Sf(x) - f(x)] = f''(x)[e(x, t)/2], \tag{14.3}$$

whenever $f''(x)$ exists.

Partial proof. That $e(x, t)$ exists involves a procedure similar to the proof of Theorem 1 in [16]. See also the remarks which conclude Section 10 above. Indeed, $e(x, t)$ is given at the very end of this paper as a specific continuous function of x and t. The requirement (14.1) is necessary, although t may be infinite.

To demonstrate (14.3), we first use (13.7), then (13.8). For $\sum_1$ we have (for sufficiently small δ and sufficiently large l)

$$\left| l \sum_1 s(\xi_j)(\xi_j - x)^2 N_j(x) \right| \leqslant \epsilon l E(x) < \epsilon B_t$$

in view of (14.2), where B_t is a liberal bound on $e(x, t)$.

Similarly, $\sum_2$ can be bounded as follows:

$$\begin{aligned} \left| l \sum_2 s(\xi_j)(\xi_j - x)^2 N_j(x) \right| &\leqslant Cl \sum_2 (\xi_j - x)^2 N_j(x) \\ &\leqslant C \frac{l}{\delta^2} \sum_2 (\xi_j - x)^4 N_j(x) \\ &\leqslant C \frac{l}{\delta^2} \sum_{-k}^{n-1} (\xi_j - x)^4 N_j(x), \end{aligned} \tag{14.4}$$

since $\delta^2 \leqslant (\xi_j - x)^2$ in $\sum_2$.

The choice (see [6], p. 117)

$$\delta = l^{-1/4}$$

leads us to the consideration of

$$l^{3/2} \sum_{-k}^{n-1} (\xi_j - x)^4 N_j(x).$$

Whether this tends to zero is not known.

A completely different approach to proving that $\sum_2$ in (14.4) tends to zero would be to consider the nature of $N_j(x)$ for those j involved in $\sum_2$. To do this would involve reference to results in [5]. Preliminary attempts suggest that this approach will eventually prove fruitful.

Our second conjecture involves the asymptotic behavior of functions possessing higher derivatives (see Bernstein [3] or Lorentz [15], p. 23).

CONJECTURE 2. *Let $f \in C^{2j}[0, 1]$ and let $Tf(x)$ be the sum of the first $2j$ terms in the Taylor expansion of f about the point x. If the hypotheses of Conjecture 1 hold, then*

$$\lim_{l \to \infty} l^j [Sf(x) - Tf(x)] = \frac{f^{(2j)}(x)}{j!} \left[\frac{e(x, t)}{2} \right]^j.$$

There is some evidence that both conjectures may be true.

By a proof exactly like theirs, we have the following converse of Conjecture 1, which, if true, would extend the Bajsanski–Bojanic Theorem [2].

THEOREM 17. *Let*

$$x_i = \frac{i}{n} \qquad (0 < i < n)$$

be the interior knots of Δ *and let*

$$0 \leqslant a < b \leqslant 1.$$

Let $f(x) \in C[0, 1]$ *and suppose*

$$\lim_{l\to\infty} l[Sf(x) - f(x)] = 0$$

for each $x \in (a, b)$.

If

$$\lim_{l\to\infty} \frac{k}{n} > 0$$

exists and if Conjecture 1 *is true, then* $f(x)$ *is linear in* (a, b).

The proof will not go through, unless

$$e(x, t) > 0 \qquad \text{for each } x \in (a, b).$$

Since $\lim k/n \neq 0$, this is the case. More specifically, the following is true:

Let $l = n + k - 1$ *and let the interior knots of* Δ *be given by*

$$x_i = i/n \qquad (0 < i < n).$$

If

$$\lim_{l\to\infty} k/n = t,$$

exists as an extended real number, then

$$\lim_{l\to\infty} lE(x) = e(x, t), \qquad (0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant \infty),$$

exists as a continuous function of x *and* t. *Moreover,*

$$e(x, t) = \begin{cases} \dfrac{(1+t)}{3t^2}\,[(2tx)^{3/2} - 3tx^2] & \left(0 \leqslant x \leqslant \dfrac{1}{2},\ 2x \leqslant t \leqslant \dfrac{1}{2x}\right) \\[2ex] \dfrac{1}{12}\,t(t+1) & \left(\dfrac{t}{2} \leqslant x \leqslant 1 - \dfrac{t}{2},\ 0 \leqslant t \leqslant 1\right) \\[2ex] \dfrac{1+t}{t}\left(x - x^2 - \dfrac{1}{6t}\right) & \left(\dfrac{1}{2t} \leqslant x \leqslant 1 - \dfrac{1}{2t},\ 1 \leqslant t \leqslant \infty\right) \end{cases}$$

and

$$e(x, t) = e(1 - x, t)(\tfrac{1}{2} \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant \infty).$$

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